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On formation of cooperative networks in presence of network effects

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Cooperation games like the *prisoners dilemma* and the *coordination game* have been used to represent many strategic interactions where individuals (also referred to as *players* or *agents*) may benefit from mutual cooperation but may have incentives to refrain from cooperation either as a dominant strategy or in response to the other player. Many situations of everyday human interaction such as opportunities of sharing resources, exchange of employment related information *etc.* have incentive structures typical of the symmetric coordination game with two pure strategy Nash equilibria; one where both agents cooperate, and one where both agents refrain from cooperation. Further, in situations where the resources or information being shared has a potentially competitive nature, it is reasonable to assume a conflict between Pareto dominance and risk dominance in the two Nash equilibria[2]. The incentive structure of the coordination game in such a context can be summarized as follows.

		Player j	
		Cooperate	Refrain
Player i	Cooperate	(x, x)	(z, y)
	Refrain	(y, z)	(w, w)

where $x > y$, $w > z$, $x > w$ and $y + w > x + z$

This paper attempts to study such interactions between multiple individuals in contexts where *network effects* of cooperation may be of importance. The attempt is to propose a model of formation of cooperative networks where the outcomes of multiple pairwise cooperation interactions determine the network of cooperation. The idea explored here is that cooperative interactions often happen in a context where the benefit or value of a link of mutual cooperation to any player may be enhanced by a factor of her own *reputation* as perceived by the other interacting player. The assumption that the perception of one's reputation by a cooperating partner enhances one's valuation of that mutually cooperative link

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[2]The Nash equilibria refinements of Pareto dominance and risk dominance were first introduced by Harsanyi and Selten (1988). The risk dominant equilibrium is one which has the largest basin of attraction; *i.e.*, the more uncertainty that a player has about the actions of the other player, the more likely she is to choose the action corresponding to the risk dominant equilibrium.

is supported by the idea that in most collaborative situations like exchange of information or resources, the quality of information that one is willing to give and the amount of resources that one is willing to share with a cooperative partner increases with one's trust on the partner. With this consideration, there may be situations where sustaining non-reciprocated cooperation is not sub-optimal for an agent because of the indirect utility or value that it generates by enhancing her reputation and trustworthiness in other interactions.

The model of cooperative network formation proposed in this paper has the following features. Agents in a finite population have the opportunity to extend cooperation to other agents. Extending and sustaining cooperation is costly; for every link of cooperation (reciprocated or non-reciprocated) that an agent i extends, a cost c_i is incurred by her. This cost is an indicator of not only the logistical costs of sharing resources or information, but also any other exogenously given factors that may increase the effort or investment that i needs to make to derive benefits from the link. In a homogeneous agents analysis, this cost is the same for all agents. As mentioned before, mutual offers of cooperation yield links of mutual cooperation that benefit both partners involved in the link while non-reciprocated cooperation yields payoffs typical of a coordination game. Further, the benefit to any agent from a mutually cooperative link is enhanced, over a base value, by a factor of her *trustworthiness*, *i.e.* her reputation as observed by her partner in that link. Agent is trustworthiness to j is simply the number of instance that agent j can observe, *through common mutual cooperative partners*, in which agent i extends cooperation. Thus information about reputation is communicated through the cooperative network itself.

This model of cooperative network formation contributes to the literature on strategic network formation, particularly to cooperative network formation. In this literature, Aumann and Myerson (1988) were the first to model network formation explicitly as a game. They proposed an extensive form game for the formation of a network (in the context of cooperative games) where pairs of agents sequentially decide whether or not to form a link between themselves. The sequence in which pairs decide is given exogenously by an ordering of pairs of agents. The formation of a link requires mutual agreement, and once formed a link cannot be broken. However, a link is not formed, can be reconsidered at a later stage of the game. Despite many appealing features, the extensive form of this game made it intractable for the purpose drawing general results. This problem was resolved by using a simultaneous move game by Myerson (1991) in which players simultaneously announce which other players they wish to be connected to, and links are formed when two agents wish to be connected to each other. While this game can be completely analysed, it faces the problem of a vast multiplicity of Nash equilibria[3]. In fact, Dutta, van den Nouweland and Tijs (1998) have analysed a large class of similar cooperative network formation games in a framework where agents first (mutually and simultaneously) decide on the links of cooperation to be formed

[3]Jackson (2005) argues that for such games (that require bilateral agreement for link formation) refinements like strong Nash equilibrium or Coalition Proof Nash equilibrium are required to narrow the set of Nash equilibria. These refinements are discussed in detail by Dutta and Mutuswami (1997).

and then negotiate over the distribution of value generated from the cooperative structure.

The model of network formation proposed here can be seen as an extension of this class of games. The added feature of this model is the possibility of forming links of *unilateral cooperation*, and that it specifies a payoff function[4] that could yield a potentially positive payoff from such links. The relevance of this proposed model comes from the fact that the payoff function has an intuitive interpretation that allows for value of any link of cooperation to be enhanced by reputation, which is determined by both bilateral and unilateral ties of cooperation that one is engaged in. In equilibrium, the value of links of cooperation is enhanced by the *embeddedness*[5] of these links in the network. This conforms to a well accepted phenomenon in social sciences, with substantial research in sociology arguing that embeddedness of social relationships enhances trust and confidence in the integrity of transactions in these relationships[6]. This also relates to the idea of *social capital*[7] and its positive impacts on cooperation in society. The next section formalises a homogeneous agents version of the model.

1 The model

Players: $N = \{1, 2, 3, ...n\}$ is a finite set of players, with $n \geq 3$.

Actions: Each player i chooses an action a_{ij} for any other player j such that

$(\forall i \in N)(\forall j \in N | j \neq i)(a_{ij} \in \{\alpha, \beta\})$

where α refers to the act of extending cooperation while β is the act of refraining from cooperating.

Strategies: Any player i's strategy is $s_i = (a_{i1}, a_{i2} ... a_{ii-1}, a_{ii+1}, ...a_{in})$. S_i is the set of all possible strategies for agent i. A strategy profile is of the form $(s_1, s_2, ...s_n)$

Link formation and neighbours:

$a_{ij} = \alpha \wedge a_{ji} = \alpha \longleftrightarrow i$ and j share a mutually cooperative tie ($\alpha - \alpha$ tie)

Or, $i \in P_j(s)$ and $j \in P_i(s)$

$a_{ij} = \alpha \wedge a_{ji} = \beta \longleftrightarrow i$ is in unilateral cooperation with j ($\alpha - \beta$ tie between i and j)

Or, $j \in E_i(s)$

$a_{ij} = \beta \wedge a_{ji} = \beta \longleftrightarrow$ non cooperative tie between i and j ($\beta - \beta$ tie)

[4]This payoff function may be interpreted in the framework of Dutta et al. (1998) as the solution or rule that specifies payoff distribution. However, this model may not be neatly analysed in their framework.

[5]The embeddedness of an edge in a network is the number of common neighbors the two endpoints have.

[6]In the literature of social networks, one of the earliest emphasis on this was laid by Granovetter (1985), who argued that "the on-going networks of social relations between people discourage malfeasance."

[7]The term social capital is defined by OECD as the networks of relationships among people who live and work in a particular society, enabling that society to function effectively.

Further, $T_{ij}(s) = P_j(s) \cap (P_i(s) \cup E_i(s))$ and $t_{ij}(s) = |T_{ij}(s)|$ is i's trustworthiness as perceived by j; which is the instances that j can observe, *through links of mutual cooperation*, in which i extends cooperation to other agents. $P_{ij}(s) = P_{ji}(s) = P_i(s) \cap P_j(s)$ is the set of common mutually cooperative neighbours that agents i and j share, with $p_{ij}(s) = p_{ji}(s) = |P_{ij}(s)|$. Note that for any strategy profile and pair of nodes, $t_{ij}(s) \geq p_{ij}(s)$

The payoff matrix of the base game (*excluding network effects*) is as follows.

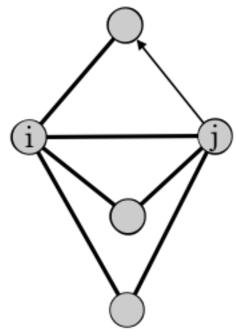
		Player j	
		α	β
Player i	α	(x, x)	(z, y)
	β	(y, z)	$(0, 0)$

where $x > y > 0 > z$ ensures that the payoffs are consistent with the coordination game and zero utility is attributed to the situation where both agents refrain from cooperating with each other. Finally, cooperation is costly to extend and sustain. For simplicity we assume that the cost of extending cooperation, whether reciprocated or non-reciprocated (unilateral) is the same. For any agent i, a cost c_i is incurred in every interaction where i cooperates. Unless agents are *altruistic*, it can be reasonably assumed that sustaining unilateral cooperation in absence of network effects of reputation is sub-optimal and $c_i > z$. For altruistic agents we assume $c_i < z$.

Then, the utility that agent i derives from strategy profile s is given by:

$$u_i(s) = \sum_{j \in P_i(s)} (t_{ij}(s) + 1)x + \sum_{j \in E_i(s)} z + \sum_{j \in N | i \in E_j(s)} y - \sum_{j \in P_i(s) \cup E_i(s)} c_i$$

Note that the value of every link of mutual cooperation is enhanced by a factor of one's own trustworthiness as perceived by the interacting partner. It is this utility enhancing impact of network based trustworthiness on links of mutual cooperation that brings in *network effects* into the model. For example, in the figure below (where darker links are links of mutual cooperation and the directed link is a link of unilateral cooperation), agent j gets communication (through links of mutual cooperation) about i's cooperation in two more instances, and hence i's reputation or (network based) trustworthiness as perceived by j is $t_{ij}(s) = 2$. On the other hand, j's reputation as perceived by j is $t_{ji}(s) = 3$.



The network effects of network based trustworthiness ensure that the incentive structure faced by any individual is not reduced to one where all interactions with other agents are independently considered.

2 Equilibria

The idea of equilibrium is used to refer to a steady state which resists any change due to internal dynamics of the model. In the context of networks, the property of equilibrium can be interpreted as an indicator of stability of the network. The equilibrium concept used here is that of the Nash equilibrium, which refers to a strategy profile from which no agent has incentives, in the form of higher payoffs, to unilaterally deviate. Formally, a strategy profile s^* is a Nash equilibrium if and only if

$$\big(\forall i \in N\big)\,(\forall s'_i \in S_i)\,\big(u_i\left(s^*_i, s^*_{-i}\right) \geq u_i\left(s'_i, s^*_{-i}\right)\big)$$

For the purpose of equilibrium analysis, changes that an agent i expects from a unilateral deviation are represented by $\triangle u_i$. Note the following:

If, from strategy profile s, agent i withdraws cooperation from an $\alpha - \beta$ link with j, then $\triangle u_i = c - z - p_{ij}(s)x$ (1)

If, from strategy profile s, agent i reciprocates cooperation to j (previously lined with a $\beta - \alpha$ link between i and j), then $\triangle u_i = -c - y + (t_{ij}(s) + 1)x + p_{ij}(s)x$ (2)

Result 1: In any Nash equilibrium profile s^*, $(\nexists i, j \in N)(a^*_{ij} = \alpha \wedge a^*_{ji} = \beta)$

That is, links of unilateral cooperation are not featured in Nash equilibria.

Proof: Suppose s^* is a Nash equilibrium profile such that $(\exists i, j \in N)(a^*_{ij} = \alpha \wedge a^*_{ji} = \beta)$.
A unilateral deviation makes i worse off. From (1), this implies:
$c - z - p_{ij}(s)x < 0 \longrightarrow c < p_{ij}(s)x + z$
A unilateral deviation makes j worse off. From (2), this implies:
$-c - y + (t_{ji}(s) + 1)x + p_{ij}(s)x < 0 \longrightarrow c > (t_{ji}(s) + p_{ij}(s) + 1)x - y$

$\longrightarrow (t_{ji}(s) + p_{ij}(s) + 1)x - y < p_{ij}(s)x + z$
$\longrightarrow (t_{ji}(s) + 1)x - y < z$

Now, $t_{ji}(s) \geq 0$ and $x > y$ implies that $(t_{ji}(s) + 1)x - y > 0$. Also, $z < 0$ by assumption. Thus Result 1 is proved by contradiction. □

The reason that unilateral cooperation is not a feature of Nash equilibria is that if any agent i finds it optimal to sustain a link of unilateral cooperation towards another agent j, it must be because of the indirect benefits that this brings. That is, i must have a sufficiently large number of common mutually cooperative partners with j so that the cost of sustaining unilateral cooperation with j is more than matched by the benefits from increased trustworthiness to all these common cooperative partners. But then j also has enough common mutually cooperative partners with i such that the benefit from reciprocating cooperation to i outweighs the cost. Thus, in any case where agent i has incentives to extend unilateral cooperation to agent j, the latter also has incentives to reciprocate cooperation to i. This means that despite being absent from Nash equilibrium strategy profiles, unilateral cooperation has a role in the best response dynamics of this model. In particular, any perturbations from equilibrium due to external disturbances[8] can be expected to trigger a best response dynamics that would rely heavily on reputation related concerns in determining convergence toward a new equilibrium.

Result 2: A complete network of mutual cooperation where $(\forall i, j \in N)(a_{ij} = \alpha \wedge a_{ji} = \alpha)$ emerges as a Nash equilibrium *if and only if* $c < (n-1)x - y$

Proving necessity: Consider a complete network of mutual cooperation. For the underlying strategy profile to be a Nash equilibrium, no arbitrary agent must have an incentive to unilaterally withdraw cooperation from any k number of links, where $n - 1 \geq k \geq 1$. That is, for all $k = 1, 2, 3....n - 1$:

$kc + ky - kx(n-1) - (n-1-k)kx < 0$
$\rightarrow c + y < [2(n-1) - k]x$
$\rightarrow c + y < x(n-1)$ (since $k \leq n - 1 \rightarrow [2(n-1) - k]x \leq (n-1)x$)

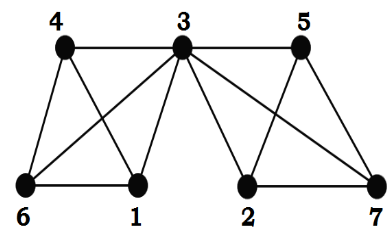
Proving sufficiency: Suppose $c < (n-1)x - y$ and the strategy profile underlying a complete network of mutual cooperation is not a Nash equilibrium. Then there exists an agent i who can profitably deviate unilaterally to withdraw cooperation from $1 \leq k \leq n - 1$ links:

$\triangle u_i = k[c + y - x(2(n-1) - k)] > 0 \longrightarrow c + y - x(2(n-1) - k) > 0$, for some $k \geq 1$
Now, $k \leq n - 1 \longrightarrow -k \geq -(n-1) \longrightarrow 2n - 2 - k \geq n - 1$
$\longrightarrow c + y - (2n - 2 - k)x \leq c + y - (n-1)x$
Further, $c + y - (n-1)x < 0 \longrightarrow c + y - (2n - 2 - k)x < 0$, for all $k \leq n - 1$
This is a contradiction with $\triangle u_i > 0$. □

[8]External perturbations may take the form of changes in costs of sustaining cooperation caused by exogenous factors.

Hence the condition $c < (n-1)x - y$ completely characterizes the complete network of mutual cooperation as a Nash equilibrium. The interpretation of the condition is simply that from a complete network of mutual cooperation, unilaterally withdrawing cooperation simultaneously from *all* partners must not be profitable to any agent. If this condition is satisfied, then there can exist multiple Nash equilibria with varying degrees of cooperation, the complete cooperation equilibrium being one of them. Example 1 illustrates the cooperative network with less than complete mutual cooperation emerging from a Nash equilibrium profile under $c < (n-1)x - y$.

Example 1 Suppose $n = 7, x = 5, y = 4, z = -1$ and $c = 6$. Then, $c < (n-1)x - y$ is satisfied as $6 < 26$. The cooperative network illustrated here (with only links of mutual cooperation shown) is supported by a Nash equilibrium strategy profile.



Consider agent 3. An arbitrary unilateral deviation can consist of withdrawing cooperation from $0 \leq k \leq 3$ agents in $\{1, 4, 6\}$ and $0 \leq m \leq 3$ agents in $\{2, 5, 7\}$ (where $k > 0 \vee m > 0$). This yields change in utility:

$\triangle u_3 = (k+m)(c+y) - kx(6-k) - mx(6-m) = 10(k+m) - 5[k(6-k) + m(6-m)]$

which is negative for any positive value of $k + m$, and zero otherwise. Thus agent 3 has no incentive to unilaterally deviate.

Consider agent 1. An arbitrary unilateral deviation can consist of a combination of withdrawing cooperation from $0 \leq k \leq 2$ agents in $\{4, 6\}$, extending cooperation to $0 \leq m \leq 3$ agents in $\{2, 5, 7\}$ and withdrawing cooperation from 3.

If 1 withdraws cooperation from 3:

$\triangle u_1 = (k+1)(c+y) - 3x(k+1) - (k+1)(2-k)x + m(z-c) = (k+1)[10 + 5(k-5)] - 7m$

which is negative for all combinations of positive values of k and m.

If i does not withdraw cooperation from 3:

$\triangle u_1 = k(c+y) - 3kx - (3-k)kx - mc + mz + mx = 10k - 5k(6-k) - 2m$

which is negative for all combinations of positive values of k and m.

Thus agent 1 has no incentive to deviate. Note that agents $1, 2, 4, 5, 6$ and 7 are symmetrically placed in this network. Thus a symmetric argument can be made to show that other agents in $\{2, 4, 5, 6, 7\}$ also have no incentive to deviate and the strategy profile underlying the given cooperation network is a Nash equilibrium. Also note that $c > x - y$ implies that mutual cooperation between any pair of agents would not have sustained in absence of network effects.

3 Conclusion

This paper presents a model of cooperative network formation that incorporates the idea of enhanced trustworthiness leading to enhanced value from cooperation and the idea of information about ones reputation being communicated through the cooperation network itself. The analysis shows that even in situations where mutual cooperation is not feasible in absence of network effects, the network effects of trustworthiness enable sustained cooperation in equilibrium. While non-reciprocated or unilateral cooperation is not a feature Nash equilibria, it plays a significant role in the best response dynamics leading to convergence to equilibria with cooperation. The homogeneous agents model explored in this paper assumes that all the agents in the finite population have equal opportunities to interact in cooperative scenarios with all other agents, and incur equal efforts to extend and sustain cooperation with everyone. This assumption may be dropped to incorporate community based heterogeneity of agents and yield segregated Nash equilibria.

A fundamental limitation of the simultaneous move formulation of the model explored in this paper is that although it does capture the incentive mechanisms at play behind the process being examined, the Nash equilibria cannot be meaningfully interpreted as outcomes emerging from a dynamic process. More importantly, while reputation plays a significant role in determining payoffs, the process can not be interpreted as an exercise in reputation building. To study an evolution process where player actively engage in reputation building and that may converge to an equilibrium, a dynamic game formulation of the model may be analysed.

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