Contemporary Issues and Ideas in Social Sciences
November 5, 2018

On the Maximal Element Rationalization of Choice Functions

Debabrata Pal[1]

1 Introduction

There are two distinct approaches to deal with consumer choice theory, namely preference relational approach and revealed preference approach. In preference relational approach, choices are interpreted in terms of a well-defined preference relation. According to this approach, the individual is assumed to choose a best alternative, whenever the set of best alternatives is nonempty. Revealed preference approach, however, does not invoke any relation between choices and preference relations. It solely deals with the choices and analyzes whether choices in different environments are consistent or not with respect to some choice consistency conditions.

An important problem in this context of choice theory is whether it is possible to construct a preference relation observing choices under different environments, such that the observed choice set is same as the set of best elements of the set with respect to that preference relation. This problem has unveiled a new aspect of choice function and is known as the *rationalizability of choice function*. The problem of rationalizability can be explained with the help of the following example.

Example:1

Let X be the set of alternatives and C be the choice function.
$X = \{x, y, z\}$; $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{x\}$, $C(\{z, y\}) = \{y\}$, $C(\{x, y, z\}) = \{x\}$

In the above choice environments x is chosen from $\{x, y\}, \{x, z\}$ and $\{x, y, z\}$, and y is chosen from $\{z, y\}$. A pertinent question in this context would be whether it is possible to construct a preference relation so that the set of best elements of every set with respect to this preference relation is same as the set of chosen elements. For this particular choice function we can construct a preference relation namely $xPyPz$ i.e., 'x is preferred to y preferred to z' which rationalizes this choice function. x is the best element of $\{x, y\}, \{x, z\}$ and $\{x, y, z\}$ according to the preference relation $xPyPz$ and it is the chosen element of the same sets as well. y is also the best element of $\{z, y\}$ and chosen element too. It is worth mentioning in

[1] Assistant Professor at Jawaharlal Nehru University. Email: debabrata@mail.jnu.ac.in; debabrata1234@gmail.com.

this context that all the choice functions are not rationalizable[2].

There are two notions of rationalizability of a choice function namely greatest (best) element rationalization(G-rationalization) and maximal element rationalization[3] (M-rationalization). A large part of the literature related to choice theory has dealt with Greatest element rationalizable choice functions and most of the results obtained in this area are related to the characterization of greatest element rationalization, where the rationalization of the choice function satisfies different rationality conditions (ordering rationalization, quasi-transitive rationalization, acyclic rationalization). A number of choice consistency conditions have been introduced in the literature by Uzawa(1957), Richter(1966), Hansson(1968), Suzumura(1976, 1983), Arrow(1959), Sen(1970) and others, which ensure the greatest element rationalizability of choice functions.

In comparison to the contribution made in the literature related to G-rationalization, literature related to M-rationalization is still not so vast. Although the concept of M- rationalization seems to have significant appeal in the domain of rational choice. In fact, the general discipline of maximization does not necessarily invoke the notion of G-rationalizability; it only requires choice set to be the set of alternatives which are no worse than others, that is precisely the set of maximal elements. Again, it may well happen that because of limited information available on some alternatives or may be because of the 'unsolved value conflict'[4] among some alternatives a preference relation over the set of alternatives turns out to be unconnected. Under such situations best element may not exist in some sets with respect to that preference relation. Hence, representation of maximizing behaviour of an individual by a G-rationalizable choice function may face a drawback. M-rationalizable choice function, on the other hand, may carry meaningful sense in some circumstances where G-rationalization fails to represent the maximizing behaviour of an individual.

Bossert et al. (2005) have provided full characterization of maximal element rationalizable choice function defined over general domain. This paper attempts to obtain the same results for the choice functions defined over full domain, by weakening some of the rationality conditions invoked by Bossert et al. (2005).

This paper is divided into four sections. Section two contains notations and definitions used in this paper. Section three briefly discusses important results of Bossert et al. (2005) on maximal-element rationalization and provides full characterization of maximal element rationalizable choice functions, quasi-transitive and acyclic maximal element rationalizable choice functions by weakening some rationality conditions (Direct irreversibility and Indirect irreversibility) under full domain. Furthermore, Bossert et al.(2005) provides existential proofs of the characterization of maximal element rationalizable choice functions, this section

[2]Consider following example:
$S = \{x, y, z\};\ C(\{x, y\}) = \{x\};\ C(\{x, z\}) = \{z\};\ C(\{z, y\}) = \{y\};\ C(\{x, y, z\}) = \{x\}$.
This choice function is not rationalizable.

[3]Definitions of greatest element and maximal element rationalization have been given in the next section 'Notations and Definitions'.

[4]See: Sen, 1970.

in addition provides full characterization of quasi-transitive maximal element rationalizable choice functions (single valued) under general domain with proper specification of the rationalization of the choice functions. Section four concludes the paper.

2 Notations And Definitions

Let X be a non-empty finite set of alternatives and 2^X be the power set of X. For a set S, $\#S$ denotes the cardinality of the set S. Let D be a nonempty collection of nonempty subsets of X, $D \subseteq 2^X - \{\emptyset\}$. A choice function C is a mapping from D to $2^X - \{\emptyset\}$, $C : D \mapsto 2^X - \{\emptyset\}$ such that $C(S) \subseteq S$ for all $S \in D$. In succeeding sections we denote D to be the domain of choice function.

Let R be a binary relation defined over X. Let I and P denote symmetric and asymmetric parts of R respectively. R defined on S is said to be
reflexive *iff* $(\forall x \in S)(xRx)$
connected *iff* $(\forall\ x, y \in S)(x \neq y \to xRy \vee yRx)$
transitive *iff* $(\forall\ x, y, z \in S)(xRy \wedge yRz \to xRz)$
quasi-transitive *iff* $(\forall\ x, y, z \in S)(xPy \wedge yPz \to xPz)$.
P-Acyclic *iff*
$(\forall\ x_1, x_2, ..., x_n \in S)(x_1Px_2 \wedge x_2Px_3 \wedge ..... \wedge x_{n-1}Px_n \to \sim x_nPx_1)$, where $n \geq 3$.

Let R_1, R_2 be two binary relations on S. Let R_1oR_2 denote *composition* of two binary relations R_1, R_2. We define R_1oR_2 as follows:
$(\forall x, y \in S)(xR_1oR_2y \leftrightarrow (\exists z \in S)(xR_1z \wedge zR_2y))$
Let $R^1 = R$, $R^2 = RoR$, $R^3 = RoR^2$, ... , $R^n = RoR^{n-1}$, where $n \in N$.
Let $\overline{T(R)}$ be the transitive closure of R.
We define $\overline{T(R)}$ as follows:
$\overline{T(R)} = \bigcup_{n=1}^{\infty} R^n$.
Let $\overline{P(R)}$ denote transitive closure of $P(R)$.

Define binary relation R_c

$$R_c = \{(x, y) \in X \times X |\ (\exists S \in D)(x \in C(S) \wedge y \in S)\}.$$

Binary relation R_c^* is defined as below:

$$R_c^* = \{(x, y) \in X \times X |\ (\exists S \in D)(x \in C(S) \wedge y \in S \setminus C(S))\}$$

x is said to be a maximal element of S with respect to R *iff* $(\forall y \in S)(\sim yPx)$.
Let $M(S, R)$ denote the set of maximal elements of the set S with respect to binary relation R.

A choice function is maximal-element rationalizable (henceforth M-rationalization) *iff* there exists a binary relation R defined over the set of alternatives such that for every set in the domain choice set is equal to the set of maximal elements of that set with respect to binary relation R, i.e.,

$(\exists R \subseteq X \times X)(\forall\ S \in D)(C(S) = M(S,R))$.

Example:2
Let $X = \{x_o, x_1, x_2\}$;
$D = \{\{x_2\}, \{x_o, x_1\}, \{x_o, x_1, x_2\}\}$.
Let $S_1 = \{x_2\}, S_2 = \{x_o, x_1\}, S_3 = \{x_o, x_1, x_2\}$.
Let C be a choice function defined over D in the following way: $C(S_1) = \{x_2\}$, $C(S_2) = \{x_o\}$, $C(S_3) = \{x_o\}$.
Now consider the following binary relation:

$$\begin{gathered} R = \{(x_o, x_1), (x_o, x_2)\} \\ M(S_1, R) = \{x_2\} = C(S_1) \\ M(S_2, R) = \{x_o\} = C(S_2) \\ M(S_3, R) = \{x_o\} = C(S_3). \end{gathered}$$

So the choice function C is M-rationalizable.

x is said to be a greatest element in S with respect to R *iff* $(\forall y \in S)(xRy)$.
Let $G(S,R)$ denote the set of greatest element of the set S with respect to binary relation R. A choice function has a greatest-element rationalization (henceforth G-rationalization) *iff* there exists a binary relation R defined over the set of alternatives such that the set of chosen elements of a set is same as the set of greatest elements of the same set with respect to binary relation R, i.e.,

$(\exists R \subseteq X \times X)(\forall\ S \in D)(C(S) = G(S,R))$.

3 Maximal element rationalization with full domain

Let D and D_f denote general domain and full domain respectively. General domain is a nonempty collection of nonempty subsets of set of alternatives. Full domain is the collection of *all* nonempty subsets of set of alternatives.

Define $A_c = \{(S,y) \mid S \in D_f \text{ and } y \in S \setminus C(S)\}$.

$F_c = \{f : A_c \mapsto X | f(S,y) \in S\}$. Few axioms need to be stated before we discuss the results. Let M, A-M and Q-M stand for maximal-element rationalization, Acyclic maximal-element rationalization and quasi-transitive maximal-element rationalization respectively.

Direct exclusion (DE): $(\forall (S,y) \in A_c)(\forall x \in X)\ (\forall T_{containing\ x} \in D)\ (f(S,y) = x \rightarrow y \notin C(T))$

Indirect Exclusion (IE): $(\forall K \in N)(\forall (S_1, x_1), (S_2, x_2), ..., (S_K, x_K) \in A_c)(\forall S_{0,containing\ x_0} \in D)((\forall k \in \{1, 2, ..., K\})(f(S_k, x_k) = x_{k-1}) \rightarrow x_K \notin C(S_0))$.

Direct irreversibility (DI): $(\forall\ x, y \in X)(\forall(S, y), (T, x) \in A_c)(f(S, y) = x \rightarrow f(T, x) \neq y)$

Indirect irreversibility (II): $(\forall K \in N)(\forall(S_0, x_0), (S_1, x_1), (S_2, x_2), ..., (S_K, x_K) \in A_c)((\forall k \in \{1, 2, ..., K\})(f(S_k, x_k) = x_{k-1}) \rightarrow (f(S_0, x_0) \neq x_K))$

DE-DI existence: If $A_c \neq \emptyset$ then there exists $f \in F_c$ satisfying DE and DI.

DE-II existence: If $A_c \neq \emptyset$ then there exists $f \in F_c$ satisfying DE and II.

IE-II existence: If $A_c \neq \emptyset$ then there exists $f \in F_c$ satisfying IE and II.

It has been shown in Bossert et al.(2005) that:
1) C is a M-rationalizable *iff* it satisfies the condition DE-DI existence.
2) C is a $A - M$-rationalizable *iff* it satisfies the condition DE-II existence.
3) C is a $Q - M$-rationalizable *iff* it satisfies the condition IE-II existence.

In this paper we have weakened the conditions DI and II under full domain. However the merit of considering the full domain may be seen in the light of arguments advanced by Arrow(1959), Sen(1971).

Now consider:

DI(2): $(\forall(S, y) \in A_c)(\forall x \in X)\ (f(S, y) = x \rightarrow (\exists\ T\ _{containing\ x,y}\ \in D)\ (x \in C(T)))$

II(2): $(\forall K \in N)(\forall(S_1, x_1), (S_2, x_2), ..., (S_K, x_K) \in A_c)(\forall x_0 \in X)((\forall k \in \{1, 2, ..., K\})(f(S_k, x_k) = x_{k-1}) \rightarrow (\exists S'_{containing\ x_0,x_K} \in D)(x_0 \in C(S')))$

The following propositions and example establish that DI(2) is weaker than DI and II(2) is weaker than II.

Proposition 1:

Let Choice function be defined on full domain, D_f and $f \in F_c$. If f satisfies Condition DI then it satisfies condition DI(2).

Proof. Let $(S, y) \in A_c$ and $f(S, y) = x$ for $x \in X$.
Since the choice function is defined over full domain we, therefore, have $\{x, y\} \in D_f$. If $x \in C(\{x, y\})$ then the conclusion follows. Suppose, $x \notin C(\{x, y\})$. By definition, we have $y \in C(\{x, y\})$. Again, the case $f(\{x, y\}, x) = x$ is prevented by the condition DI. We, thus, have $f(\{x, y\}, x) = y$. (1)
Condition DI further implies $f(\{x, y\}, x) \neq y$ which in view of (1) leads to a contradiction. □

Proposition 2:

Let Choice function be defined on full domain, D_f and $f \in F_c$. If f satisfies Condition II then it satisfies condition II(2).

Proof. Proof is analogous to proposition 1. □

Example:3

$X = \{x, y, z\}$ and $\Delta_x = \{\{x\} \mid x \in X\}$.

$D_f = \{\{x, y, z\}, \{x, y\}, \{x, z\}, \{y, z\}\} \cup \Delta_x$.

$S_1 = \{x, y, z\}, S_2 = \{x, y\}, S_3 = \{x, z\}, S_4 = \{y, z\}$.

$C(\{x, y, z\}) = \{y, z\},\ C(\{x, y\}) = \{x\},\ C(\{x, z\}) = \{x\},\ C(\{y, z\}) = \{y\}$.

$A_c = \{(S_1, x), (S_2, y), (S_3, z), (S_4, z)\}$.

(1).......$f(S_1, x) = y,\ f(S_2, y) = x,\ f(S_3, z) = x,\ f(S_4, z) = y$.

(2).......$f(S_1, x) = z,\ f(S_2, y) = x,\ f(S_3, z) = x,\ f(S_4, z) = y$.

It is only required to explain that the two functions in the above example satisfy II(2) but violate DI. Since the condition II is stronger than DI and so is the case between II(2) and DI(2), violation of DI would imply violation of II and satisfaction of II(2) would imply satisfaction of DI(2).

It is easy to verify that both the functions satisfy II(2). The first function violates DI due to $f(S_1, x) = y,\ f(S_2, y) = x$. The second violates the same due to $f(S_1, x) = z,\ f(S_3, z) = x$.

Now we can introduce new existential conditions as below:

DE-DI(2) existence: If $A_c \neq \emptyset$ then there exists $f \in F_c$ satisfying DE and DI(2).

DE-II(2) existence: If $A_c \neq \emptyset$ then there exists $f \in F_c$ satisfying DE and II(2).

IE-II(2) existence: If $A_c \neq \emptyset$ then there exists $f \in F_c$ satisfying IE and II(2).

Theorem 1, 2 and 3 reproduce the same results obtained by Bossert et al. (2005) under full domain (i.e., $D = D_f$) using the weaker rationality conditions.

Theorem 1:
C defined on full domain, is M-rationalizable iff *it satisfies the condition DE-DI(2) existence.*

Proof:

Let R be M-Rationalization of C.

If $A_c = \emptyset$, then condition DE-DI(2) existence is trivially satisfied.

Let $A_c \neq \emptyset$.

Define a function f:$A_c \mapsto X$.

let$(S, y) \in A_c$ (as $A_c \neq \emptyset$).

$(S, y) \in A_c \rightarrow S \in D$ and $y \in S \setminus C(S)$, (by construction).

R is M-Rationalization $\rightarrow (\exists\ x\ \in S)$ (xPy).

Let us define $f(S, y) = x$.

Hence $f(S, y) \in S$, $\forall (S, y) \in A_c$.

We need to show that f satisfies the conditions DE and DI(2).

Condition DE:

Let $f(S, y) = x$, $T \in D$ and $x \in T$.

By definition of f, we have xPy.

Since R is M- Rationalization of C, therefore $y \notin C(T)$.

Condition DI(2):

Let $f(S, y) = x$

$f(S, y) = x \to xPy$, (By definition of f).

Since Ω includes all doubleton sets,

$\therefore \{x\} = C(\{x, y\})$, (as R is M-Rationalization of C).

$\therefore$ Condition DI(2) is satisfied.

Now, if $A_c = \emptyset$, then any relation R with $P(R) = \emptyset$ is M-Rationalization of C.

If $A_c \neq \emptyset$, then there exists f satisfying conditions DE and DI(2).

Define $R = \{(f(S, y), y) | (S, y) \in A_c\}$.

Let $x \in C(S)$.

Suppose $(\exists y \in S)$ (yPx).

$yPx \to f(T, x) = y$, for some $T \in D$ and $x \in T \setminus C(T)$.

$f(T, x) = y \to x \notin C(S)$. (by condition DE).

This is a contradiction.

$\therefore x \in M(S, R)$.

$\therefore C(S) \subseteq M(S, R)$.

let $x \notin C(S)$.

$\therefore f(S, x) = y$, for some $y \in S$.

$f(S, x) = y \to yRx$.

Suppose xRy.

$xRy \to f(T, y) = x$ for some $T \in D$.

$$f(T, y) = x \to (\exists V_{containing\ x,y} \in D)(x \in C(V)) \quad (1.1)$$

. Again, $f(S, x) = y \to (\forall T_{containing\ y} \in D)(x \notin C(T))$ (by DE). (1.2)

(1.1) and (1.2) together lead to a contradiction.

$\therefore \sim xRy$.

$\therefore yPx$.

$yPx \to x \notin M(S, R)$.

$\therefore M(S,R) \subseteq C(S)$.

Theorem 2:
C defined on full domain, is Q-M rationalizable iff *it satisfies the condition IE-II(2) existence.*

Proof:
Let R be Q-M Rationalization of C.

If $A_c = \emptyset$, then condition IE-II(2) existence is trivially satisfied.

Let $A_c \neq \emptyset$.

Define a function $f{:}A_c \mapsto X$.

Let$(S, y) \in A_c$ (as $A_c \neq \emptyset$).

$(S, y) \in A_c \rightarrow S \in D$ and $y \in S \backslash C(S)$ (by construction)

R is Q-M Rationalization $\rightarrow$ ($\exists$ x $\in S$) (xPy)

Let us define $f(S, y) = x$.

Hence $f(S, y) \in S$, $\forall (S, y) \in A_c$.

We need to show that f satisfies the conditions IE and II(2).

Condition IE:

Suppose $K \in N \ \wedge \ (S_1, x_1), (S_2, x_2), ..., (S_K, x_K) \in A_c \wedge S_0 \in \Omega \wedge x_0 \in S_0$ such that $(\forall k \in \{1, 2, ..., K\})(f(S_k, x_k) = x_{k-1})$.

$(x_{k-1}, x_k) \in P(R), \forall k \in \{1, 2....K\}$ (by definition).

$\rightarrow (x_0, x_K) \in P(R)$ (by quasi-transitivity of R).

$\therefore x_K \notin C(S_0)$. (as R is Q-M rationalization of C).

Condition II(2):

Suppose $K \in N \wedge (S_1, x_1), (S_2, x_2), ..., (S_K, x_K) \in A_c) \wedge x_0 \in X$ such that $(\forall k \in \{1, 2, ..., K\})(f(S_k, x_k) = x_{k-1})$.

$(x_{k-1}, x_k) \in P(R), \forall k \in \{1, 2....K\}$ (by definition)

$\rightarrow (x_0, x_K) \in P(R)$ (by quasi-transitivity of R)

$\rightarrow C(\{x_0, x_K\}) = \{x_0\}$ (as R is Q-M rationalization of C).

Now, if $A_c = \emptyset$, then any relation R with $P(R) = \emptyset$ is Q-M Rationalization of C.

If $A_c \neq \emptyset$, then there exists f satisfying conditions IE and II(2).

Let R be the transitive closure of $\{(f(S, y), y) | (S, y) \in A_c\}$

Let $S \in D,$ and $x \in S$
Let $x \in C(S)$

Suppose $(\exists y \in S)$ (yPx)

$\rightarrow (\exists K \in N)(\exists (S_1, x_1), (S_2, x_2), ..., (S_K, x_K) \in A_c)(x_0 = y \wedge x_K = x \wedge (\forall k \in \{1, 2, ..., K\})$

$(f(S_k, x_k) = x_{k-1}))$
$\rightarrow x \notin C(S)$ (by condition IE) (2.1)
Again, $x \in C(S)$ (by assumption) (2.2)
(2.1) $\wedge$ (2.2) lead to a contradiction.

Let $x \notin C(S)$

Let $f(S, x) = y$

$\therefore (y, x) \in R$.
$(y, x) \in R \rightarrow (\forall T_{containing\ y} \in D)(x \notin C(T))$ (by DE implied by condition IE) (2.3)
Suppose $(x, y) \in R$.

$\rightarrow (\exists K \in N)(\exists (S_1, x_1), (S_2, x_2), ..., (S_K, x_K) \in A_c)((S_0, x_0) = (S, x) \wedge x_K = y \wedge (\forall k \in \{1, 2, ..., K\})(f(S_k, x_k) = x_{k-1}))$ (2.4)
(2.4) $\rightarrow (\exists S'_{containing\ x,y} \in D)(x \in C(S'))$ (by II(2)) (2.5)
(2.3)$\wedge$ (2.5) lead to a contradiction.

$\therefore (y, x) \in P(R)$.

$\rightarrow x \notin M(S, R)$.

Further notice, R is transitive as it is a transitive closure which implies that R is quasi-transitive too.

Theorem 3:
C defined on full domain, is A-M rationalizable iff *it satisfies the condition DE-II(2) existence.*

Proof:
Let R be A-M rationalization of C.

If $A_c = \emptyset$, then condition DE-II(2) existence is trivially satisfied.

Let $A_c \neq \emptyset$.

Define a function f:$A_c \mapsto X$.

Let$(S, y) \in A_c$ (as $A_c \neq \emptyset$).

$(S, y) \in A_c \rightarrow S \in D$ and $y \in S \setminus C(S)$ (by construction).

R is A-M rationalization $\wedge y \notin C(S) \rightarrow (\exists\ x\ \in S)$ (xPy).

Let us define $f(S, y) = x$.

Hence $f(S, y) \in S$, $\forall (S, y) \in A_c$.

We need to show that f satisfies conditions DE and II(2).

We have already shown the satisfaction of condition DE in Theorem 1.

Condition II(2):

Suppose $K \in N \wedge (S_1, x_1), (S_2, x_2), ..., (S_K, x_K) \in A_c \wedge x_0 \in X$ such that $(\forall k \in \{1, 2, ..., K\})(f(S_k, x_k) = x_{k-1})$.

$(x_{k-1}, x_k) \in P(R), \forall k \in \{1, 2....K\}$ (by definition)

$\rightarrow (x_K, x_0) \notin P(R)$

$\rightarrow x_0 \in C(\{x_0, x_K\})$ (as R is A-M rationalization of C).

The sufficiency part of the proof we have done in Theorem 1. We shall show only that $R = \{(f(S, y), y) \mid (S, y) \in A_c\}$ is P-Acyclic.

Let $x_0 P x_1 \wedge x_1 P x_2 \wedge x_2 P x_3 \wedge .... \wedge x_{K-1} P x_K$
$\rightarrow (\forall k \in \{1, 2....K\})(f(S, x_k) = x_{k-1})$ where $K \in N$
$\rightarrow (\exists V \in D_{containing\ x_0, x_K})(x_0 \in C(V))$ (by condition II(2)) (3.1)
Suppose $x_K P x_0$

$x_K P x_0 \rightarrow f(S', x_0) = x_K$ *for some* $S' \in D$

$f(S', x_0) = x_K \rightarrow (\forall S \in D)(x_K \in S \rightarrow x_0 \notin C(S)).$ (3.2)
$(3.1) \wedge (3.2)$ lead to a contradiction.

$\therefore (x_K, x_0) \notin P(R).$

It is important to note that the characterization of maximal element rationalizability, irrespective of the nature i.e., acyclic maximal element rationalizability or quasi-transitive maximal element rationalizability, as provided by Bossert et al.(2005) establishes the necessary and sufficient conditions for the existence of maximal element rationalization of a choice function but does not specifically talk about the binary relation which rationalizes the choice function. The following theorem provides a necessary and sufficient condition for quasi-transitive rationalization of a single-valued choice function, and provides a binary relation which is the quasi-transitive rationalization of the choice function.

Lemma 1:
If R is Q-M rationalization of C' where C' is a single valued choice function defined on general domain, then
$(\forall x, y \in X)(xP(R_c)y \rightarrow xP(R)y).$

Proof:
Let R be Q-M rationalization of C' and $x, y \in X$.

Suppose $xP(R_c)y$.

$xP(R_c)y \leftrightarrow xR_c y \wedge \sim yR_c x.$

$xR_c y \rightarrow (\exists S' \in D)\ (x \in C'(S') \wedge y \in S').$ (i)
$\sim yR_c x \rightarrow (\forall S \in D)\ (y \notin C'(S) \vee x \notin S).$ (ii)
$i \wedge ii \rightarrow (x \in C'(S') \wedge y \in S' \backslash C'(S')).$

$y \notin C'(S') \rightarrow (\exists z \in S')(zP(R)y)$ ($\because$ R is Q-M rationalization of C').

Case 1: $z \in C'(S')$

Since C' is single valued function, therefore, $xP(R)y$.

Case 2: $z \notin C'(S')$.

$z \notin C'(S') \rightarrow (\exists w_1 \in S')(w_1 P(R) z)$

$w_1 P(R) z \wedge z P(R) y \rightarrow w_1 P(R) y$ ($\because$ R is Q-M rationalization of C').

Let $w_1 \notin C'(S')$.

$w_1 \notin C(S') \rightarrow (\exists w_2 \in S')(w_2 P(R) w_1)$.

$w_2 P(R) w_1 \wedge w_1 P(R) y \rightarrow w_2 P(R) y$.

This would continue till we get $w_n \in C'(S')$ for $n \in N$,such that

$w_n P(R) w_{n-1} \wedge w_{n-1} P(R) y \rightarrow w_n P(R) y$. since C' is single valued function, therefore, x P(R)y.

Theorem 4:

C' *defined on general domain, is Q-M Rationalizable* iff

(1) $(\forall S \in D)(\forall y \in S)(y \notin C'(S) \rightarrow (\exists z \in S)(z P(R_c) y))$.

(2) $(\forall x, y \in X)((x, y) \in \overline{P(R_c)} \rightarrow (\forall S \in D)(x \in S \rightarrow y \notin C'(S))$.

Proof:

Let R be the Q-M rationalization of C'. We need to show that both the conditions are satisfied.

Satisfaction of condition 1:

Let $S \in D,\ y \in S \backslash C'(S)$

$y \notin C'(S) \rightarrow (\exists z \in S)(z P(R) y)$

$$z P(R) y \rightarrow (\forall S_{containing\ z,y} \in D)(y \notin C'(S)) \text{ } (\because \text{R is Q-M rationalization of } C'). \tag{4.1}$$

Case 1: $z \in C'(S)$

$z \in C'(S) \wedge (4.1) \rightarrow z P(R_c) y$.

Case 2: $z \notin C'(S)$.

$z \notin C'(S) \rightarrow (\exists w_1 \in S)(w_1 P(R) z)$

$w_1 P(R) z \wedge z P(R) y \rightarrow w_1 P(R) y$ ($\because$ R is Q-M rationalization of C').

Let $w_1 \notin C'(S)$.

$w_1 \notin C'(S) \rightarrow (\exists w_2 \in S)(w_2 P(R) w_1)$.

$w_2 P(R) w_1 \wedge w_1 P(R) y \rightarrow w_2 P(R) y$.

This would continue till we get $w_n \in C'(S)$ for $n \in N$, such that

$w_n P(R) w_{n-1} \wedge w_{n-1} P(R) y \rightarrow w_n P(R) y$.

$$w_n P(R) y \rightarrow (\forall S_{containing\ w_n,y} \in D)(y \notin C'(S)) \tag{4.2}$$

$(4.2) \wedge w_n \in C'(S) \rightarrow w_n P(R_c) y$

Satisfaction of Condition 2:

Let $(x, y) \in \overline{P(R_c)}$

$(x, y) \in \overline{P(R_c)} \rightarrow (\exists z_1, z_2, ..., z_{n-1} \in X)(xP(R_c)z_1, z_1P(R_c)z_2, ..., z_{n-1}P(R_c)y)$, where $n \in N$

$\rightarrow (xP(R)z_1, z_1P(R)z_2, ..., z_{n-1}P(R)y)$ (by lemma 1)

$\rightarrow xP(R)y$ ($\because$ R is Q-M rationalization of C')

$\rightarrow (\forall S \in D)(x \in S \rightarrow y \notin C'(S))$.

Let R' be the transitive closure of $P(R_c)$.

Let $x \in M(S, R')$.

Suppose $x \notin C'(S)$

$x \notin C'(S) \rightarrow (\exists z \in S)(zP(R_c)x)$ (by condition 1)

$\therefore x \notin M(S, R')$

This is a contradiction.

$\therefore x \in C'(S)$

Let $x \in C'(S)$.

Suppose $(\exists y \in S)(yP(R')x)$.

$\rightarrow (y, x) \in \overline{P(R_c)}$.

$\rightarrow x \notin C'(S)$ (by condition 2)

This is a contradiction.

$\therefore x \in M(S, R')$

Hence $x \in C(S)$.

4 Conclusion

The literature on rationality and the choice consistency conditions is mostly centered around the notion of G-rationalization, whereas the concept of M-rationalization in rational choice theory has received fairly less attention.

As noted before, the concept of M-rationalizability has its own merit especially in situations where due to limited information on some of the alternatives or due to unsolved value-conflict among alternatives preference relation turns out to be incomplete and consequently may not generate best element for some feasible sets of alternatives. The idea of M-rationalization carries meaningful sense in such occasions for representing the maximizing behaviour of an individual.

Since domain restriction (full domain) on choice function carries plausible meaning as argued by Sen(1971) and Arrow(1959), retaining Bossert et al.(2005)'s results in full domain in presence of weaker conditions (Theorem 1, 2 and 3) seems worth pursuing. Theorem 4, however may be of more importance, as it characterizes Q-M rationalizable single valued choice functions under general domain. The proof provides a binary relation which is the rationalization of choice function; construction of such binary relation is absent in Bossert et al.(2005).

Acknowledgement
I want to thank Prof. Satish K. Jain, Dr. Taposik Banerjee and Dr. Papiya Ghosh for their helpful comments and suggestions.

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