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On the k^{th} Best Quasi-Transitive Rationalizability of Choice Functions

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In his 1984 Presidential address of the Econometric Society, Amartya Sen argued against the standard practise of imposing axioms of "internal consistency" on the choice function that require consistency between different parts of a choice function, without taking into account anything other than the choice function.[2] He points out the difference between internal consistency that is *a priori imposed* on the choice function and the one that *entails* from the choice behaviour. While arguing against the former, Sen says that there is no internal way by which one can determine whether or not the choice behaviour is consistent. This issue arises because of the presumption that the *act* of choice can be viewed as a stand-alone action. He draws attention to different circumstances under which an internally inconsistent choice behaviour would seem to be perfectly consistent.[3] Individuals might be operating under some norms which might be influencing their act of choice, benevolence might lead them to alter their choice behaviour or they could simply be following

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[2]SeeSen (1993).

[3]See Sen (1997) also.

what society perceives as proper or good behaviour. To emphasise this we quote Sen's famous example. Suppose a person has been offered some cake and has to choose from the different pieces available. Let x, y and z be three pieces of different sizes such that z is bigger than y which is bigger than x. It is assumed that the individual has no difficulty assessing the difference between them. Suppose on being offered x and y, she chooses x and from x, y and z she decides to pick up y. If this person is fond of cakes and would like to choose the biggest piece, then her behaviour is indeed inconsistent. Alternatively, suppose she does not want to appear greedy and is operating under a norm of never choosing the largest piece. Keeping this in mind and that the size of z is the biggest and x the smallest, her behaviour is now perfectly rational.

In the above example, it was assumed that the individual has no difficulty assessing the difference between the size of the slices. However, this may not always hold. Suppose a person has been offered some cold drink and has to choose from different glasses of the same drink. Say there are three glasses with 150ml, 140ml and 130ml of the same cold drink. Further suppose that the individual can differentiate in the volume of the drink only if there is a minimum difference of 20ml. Now, say on being offered 150ml and 140ml, she picks the glass with 150ml of cold drink; on being offered 140ml and 130ml, she picks 140ml. But when she is offered all three, she picks 130ml. If this person likes the drink and would like to pick the biggest glass, then her behaviour is inconsistent. However, if she does not want to appear greedy and, as before, is operating under a norm of never choosing the biggest glass, her behaviour now seems consistent. If she cannot differentiate between 150ml and 140ml, she could have randomly picked 150ml. Similarly, she randomly picked 140ml when offered 140ml and 130ml. But when all three were offered, she could differentiate between 150ml and 130ml and thus, picked 130ml. This inability to differentiate between the alternatives violates transitivity and thus, other consistency conditions need to be explored. These examples perfectly depict how external references are important while rationalizing choice behaviour. As it is obvious, this kind of choice behaviour will violate the standard axioms of internal consistency and as long as economic theories rely on the conventional axiomatic structure, new conditions need to be devised to include more possibilities.

Inspired by Sen's example, Baigent and Gaertner (1996) have provided characterisations for choice behaviour that picks the second largest element if there is a uniquely largest; otherwise, the largest elements are picked. Gaertner and Xu (1999a) provide characterisation conditions for median rationalizability. These papers have also hinted at further possible developments towards the k^{th} best.[4] The characterisation for choosing the second best has been given by Banerjee (2008a,b). Banerjee (2009b) provides a characterisation for the k^{th} best ordering rationalization in full domain. In this paper we will discuss the intuition behind the conditions of k-rationalizability by a reflexive, connected and quasi-transitive rationalization. We will be working with the framework developed by Banerjee (2008a,b, 2009b). We would first be looking at a choice behaviour where second best elements are chosen whenever available. Later, we would like to generalise it to the cases where

[4]See Gaertner and Xu (1997, 1999b) also.

k^{th} best elements are chosen when they exist.

Let us first define second best elements. Let B be the set of best elements of the set S. If we remove the elements of the set B from set S, we are left with the reduced set $S - B$. Now, we find best elements of this set $S - B$. The best elements of the set $S - B$ are the second best elements of the set S as they are the best elements of the set in the absence of the elements of set B. Whenever second best elements are available, they should be the ones chosen. If second best elements are not available, the first best elements must be chosen. A choice function is second best rationalizable (or *2-rationalizable*) if we can generate a binary relation such that if second best elements of that relation are chosen, whenever available, same choices are obtained. On similar lines we can obtain the third best elements of the set by eliminating the first best and the second best elements of the set and finding the best elements from the remaining set. A more general form of this kind of choice behaviour would be where k^{th} best elements are chosen, whenever they exist, where k is a positive integer ($k \geq 2$). We can arrive at k^{th} best elements by picking the best elements after similarly eliminating consecutive sets of best elements up till $(k-1)^{th}$ best elements. That is to say, we remove the first best, second best, ..., $(k-1)^{th}$ best elements and find the best elements from the remaining set to obtain the k^{th} best elements of the set. A choice function is k^{th} best rationalizable (or *k-rationalizable*) if we can generate a binary relation such that if k^{th} best elements of that relation are chosen, whenever available, same choices are obtained. If, however, k^{th} best elements are not available, $(k-1)^{th}$ best elements are chosen and if $(k-1)^{th}$ best elements are also not available, $(k-2)^{th}$ best are chosen and so on. Note here that if k^{th} best elements are not available, the chosen elements are the worst elements of the set.

We will first look at the case where a choice function is 2-rationalizable with full domain. If we have a set without any best elements, it would lead to the choice set being empty. Therefore, the assumption of full domain and the assumption that the choice sets are non-empty guarantee us the existence of best elements in every set. If there are no second best elements in a set, given full domain, it must be that all elements in the set are best elements. Therefore, if we see a best element in the choice set, we know that there are no second best elements in that set. In that case, the choice set must be equal to the set itself. If all elements are chosen, because all of them are best, then for every non-empty subset of the set, their choice sets must be the entire set. This is the idea behind our first axiom (A1). If not all elements are best elements then obviously there are some second best elements and the choice set would be the set of these second best elements. The fact that these elements are second best implies for every second best element there must be some element which is preferred to it. These preferred elements ought to be the best elements of the set. Hence, whenever non-best elements exist, the choice set will consist of only second best elements and for each of these chosen elements, a preferred element exists and all elements that are preferred to it are best elements. This is what our second axiom (A2) requires. Together these two axioms characterise a 2-rationalizable choice function under full domain.[5] Reflexivity and connectedness are obtained as a by-product of full domain. Here, we introduce our third axiom (A3) that

[5]See Banerjee (2009a).

would ensure us quasi-transitivity. A3 requires that whenever there is a non-best element, there must exist a best element that is preferred to it. Notice that this axiom does not demand that every best element be preferred to any non-best element, it merely requires that for any non-best element there must be some best element that is preferred to it.[6] Together these three axioms constitute necessary and sufficient conditions for the existence of a reflexive, connected and quasi-transitive 2-rationalization of the choice function with full domain.

Continuing with the assumption of full domain, we now move on to the characterisation of a k-rationalizable choice function, where k is a positive integer. If $k = 1$, we are back to the standard case of best rationalizability and $k = 2$ brings us back to the case of second best rationalizability. Therefore, we are only interested in cases where $k \geq 3$. Consider a choice function which is k-rationalizable. Now let there be a set that contains a j^{th} best element in it, where $j > 1$. If this element is j^{th} best, it is obvious that there must be a $(j-1)^{th}$ best element that is preferred to this j^{th} best element and for this $(j-1)^{th}$ best element, there must exist a $(j-2)^{th}$ best element that is preferred to it and so on. Thus, for the existence of a j^{th} best element, there needs to be a sequence of distinct j number of elements in the set such that in each consecutive pairs of elements, the first element is preferred to the immediate next element and the last element being the j^{th} best element. That is to say, the element that is placed first in this sequence is preferred to the second element, second element is preferred to the third, and so on and

[6]If we require that every best element is preferred to any non-best element then we obtain transitivity. See Banerjee (2009a).

lastly you have the $(j-1)^{th}$ element preferred to the j^{th} element where the j^{th} element is the j^{th} best element. Since we began by saying that a j^{th} best element exists, we know that the length of such a sequence is j. If, however, a j^{th} best element does not exist in the set then there cannot be any such sequence of distinct j number of elements present in the set. If at all there is a sequence, it must be of a length less than j. There may be many such sequences in a set. We call the length of the longest of such sequences as the *order of the set*. If the length of the longest sequence in a set S is l then the order of the set S is l, where l is a positive integer. Then we know there is an l^{th} best element present in the set and there does not exist any $(l+1)^{th}$ best element in the set. The length of such a sequence tells us the number of distinct consecutive preference levels present in the set. If we are considering k-rationalizability in full domain, we can observe all pair-wise choices and in every such pair-wise choice we know that the second best element will be chosen whenever there exists one. Otherwise both elements are chosen. Hence, we can easily form a sequence of the kind described above and the order of the set can be effortlessly determined.

Suppose the choice function is k-rationalizable with $k \geq 3$. The chosen elements then must be the k^{th} best elements, whenever they exist. Let us consider the case where k^{th} best elements are chosen. In that case there must exist some preferred elements in the set. Evidently this set of preferred elements cannot have any k^{th} best element in it. There must be a sequence of k consecutively preferred elements of which the last and the least preferred element is the k^{th} best element. For a quasi-transitive k-rationalization, ignoring the last element (that is the k^{th} best element), the remaining $k-1$

elements must be there in the set of preferred elements constituting $k-1$ distinct and consecutive preference levels. Hence, the order of this set of preferred elements must be $k-1$. Therefore, whenever the order of a set is j and $j \geq k$, we know that k^{th} best elements exist, then for every chosen element the order of the preferred set must be exactly equal to $k-1$. This requirement has been formalised as axiom A4. Next we see what happens when a k^{th} best element does not exist. In the absence of k^{th} best elements, we search for the $(k-1)^{th}$ best elements and if they are also not available, we look for the $(k-2)^{th}$ best elements, and so on. In such a scenario, irrespective of which elements we end up choosing, they are essentially the worst of the set. Therefore, whenever the order of a set is $j < k$, the order of the set of preferred elements must be $j-1$ thereby ensuring that the worst elements of the set are chosen. This is our axiom A5. Together axioms A4 and A5 characterise a choice function that is k-rationalizable under full domain by a quasi-transitive k-rationalization. Full domain once again ensures that this k-rationalization is reflexive and connected. Similar characterisations for a choice function to be k-rationalizable with general domain remains largely unexplored. Needless to say, the area of k^{th} best offers a lot to be explored. Any further result in this context would be of great significance.

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